

ON THE INFINITE-DIMENSIONAL HIDDEN SYMMETRIES. I. INFINITE DIMENSIONAL GEOMETRY OF q_R -CONFORMAL SYMMETRIES.

DENIS V. JURIEV

“Thalassa Aitheria” Research Center for Mathematical Physics and Informatics,
ul.Miklukho-Maklaya 20-180, Moscow 117437 Russia.

E-mail: denis@juriev.msk.ru

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ABSTRACT. The infinite dimensional geometry of the q_R -conformal symmetries (the hidden symmetries in the Verma modules over $\mathfrak{sl}(2, \mathbb{C})$ generated by the spin 2 tensor operators) is discussed.

This paper opens the series of articles supplemental to [1], which also lie in lines of the general ideology exposed in the review [2]. The main purpose of further activity, which has its origin and motivation presumably in the author’s applied researches [3] on the interactively controlled systems (i.e. the controlled systems, in which the control is coupled with unknown or uncompletely known feedbacks), is to explicate the essentially *infinite-dimensional* aspects of the hidden symmetries, which appear in the representation theory of the finite dimensional Lie algebras and related algebraic structures. The series is organized as a sequence of topics, which illustrate this basic idea on the simple and tame examples without superfluous difficulties and details as well as in the series [1]. Many of objects, which will appear, are somehow related to ones discussed previously. However, the material will be treated more geometrically, presumably, from the points of view of the infinite dimensional geometry (cf.[4]), an infinite dimensional version of the nonlinear geometric algebra (cf.[5]) and the infinite dimensional noncommutative geometry (cf.[6]). The intent reader will see a lot of ideological similarities between the subject of the paper and the infinite dimensional geometric picture for a second quantized string (see [7] and numerous refs wherein).

1. UNITARY \mathcal{HS} -PSEUDOREPRESENTATIONS OF
 $\widetilde{\text{Diff}}_+(S^1)$, INFINITE DIMENSIONAL GEOMETRY OF
 q_R -CONFORMAL SYMMETRIES AND THE RELATED TOPICS

Definition 1A. A unitary \mathcal{HS} -pseudorepresentation of the Lie group G in the Hilbert space H is a homomorphism of G into $\mathbf{U}(H)/\mathbf{U}_0(H)$, where $\mathbf{U}(H)$ is the

group of all unitary operators in H and $\mathbf{U}_0(H)$ is its subgroup of operators U such that $1 - U$ is a Hilbert–Schmidt operator.

The motivation for the choice of the name “ \mathcal{HS} –pseudorepresentation” is an analogy with the constructions of the pseudodifferential calculus [8], the abbreviation \mathcal{HS} means that an admissible deviation from the “true” representations belongs to the class \mathcal{HS} of the Hilbert–Schmidt operators (one may also consider the operator ideals of compact or trace-class operators instead of \mathcal{HS} , it allows to generalize the constructions to the linear topological spaces). It is rather convenient to consider the pullbacks $\tilde{\pi} : G \mapsto \mathbf{U}(H)$ instead of the \mathcal{HS} –pseudorepresentations π themselves. The pullback is called *monoassociative* iff $\tilde{\pi}(g_{t+s}) = \tilde{\pi}(g_t)\tilde{\pi}(g_s)$ for any one–parameter subgroup $\{g_t\}$ of G . The pullback $\tilde{\pi}$ obeys the following claim:

$$\tilde{\pi}(g_1)\tilde{\pi}(g_2) \equiv \tilde{\pi}(g_1g_2) \pmod{\mathcal{HS}}$$

for g_1, g_2 from G .

Definition 1B. Two unitary \mathcal{HS} –pseudorepresentations π_1 and π_2 of the Lie group G in the Hilbert spaces H_1 and H_2 are called *equivalent* iff there exists an isomorphism $S : H_1 \mapsto H_2$ of the linear spaces such that for any element g of the group G there exists an element U_g of $\mathbf{U}_0(H)$ such that

$$\tilde{\pi}_2(g) = U_g S \tilde{\pi}_1(g) S^{-1},$$

where $\tilde{\pi}_i$ are the mappings of G into $\mathbf{U}(H_i)$, which are the pullbacks of the homomorphisms π_i alongside the natural projections $\varepsilon_i : \mathbf{U}(H_i) \mapsto \mathbf{U}(H_i)/\mathbf{U}_0(H_i)$.

The infinitesimal counterpart of the notion of the unitary \mathcal{HS} –pseudorepresentation of the Lie group G is one of the \mathcal{HS} –projective representation of the Lie algebra \mathfrak{g} [1:Topic 10]. The idea of the unitary \mathcal{HS} –pseudorepresentation is ideologically very closed to one of the quasirepresentation of A.I.Shtern [9] (A.I.Shtern also defined the pseudorepresentations in a way analogous to the definition of a pseudocharacter, however, his definition differs from ours. Below we shall not discuss A.I.Shtern’s pseudorepresentations).

Let us now consider the \mathcal{HS} –projective representations τ_h of the Witt algebra in the unitarizable Verma modules V_h (h is the highest weight) over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Such representations realize the generators of the Witt algebra as tensor operators of spin 2 in the Verma modules or, otherwords, as the q_R –conformal symmetries [3]. The \mathcal{HS} –projective representations τ_h may be extended to the \mathcal{HS} –projective representations τ_h of the algebra $\text{Vect}(\widehat{S^1})$ of the smooth vector fields on a circle in the suitable Gårding space. Let $\widehat{\text{Diff}}_+(S^1)$ be the universal covering of the group $\text{Diff}_+(S^1)$ of the orientation preserving diffeomorphisms of a circle S^1 .

Theorem 1. *The \mathcal{HS} –projective representations τ_h of the Lie algebra $\text{Vect}(\widehat{S^1})$ are exponentiated to the unitary \mathcal{HS} –pseudorepresentations T_h of $\widehat{\text{Diff}}_+(S^1)$ in the Hilbert spaces H_h , which are the completions of the unitarizable Verma modules V_h over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.*

Remark 1. There exists a uniquely defined continuous monoassociative pullback \tilde{T}_h of the unitary \mathcal{HS} –pseudorepresentation T_h such that the infinitesimal operators for

the generators of $\text{Vect}(S^1)$ coincide with one of the \mathcal{HS} -projective representation of the Witt algebra in the Verma module over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ by the tensor operators of spin 2 [1:Topic 10].

Remark 2. The theorem may be generalized to the non-unitarizable Verma modules if one changes the class \mathcal{HS} of the Hilbert-Schmidt operators to the class of all compact operators.

Let $\mathcal{M} = \text{Diff}_+(S^1)/S^1$ be the flag manifold for the Virasoro–Bott group [10;4].

Corollary. The monoassociative pullback \tilde{T}_h defines an imbedding $\epsilon : \mathcal{M} \mapsto \mathbf{P}(H_h)$ of the infinite-dimensional manifold \mathcal{M} into the projective space $\mathbf{P}(H_h)$ as an orbit \mathcal{O} of the highest vector.

Let us consider the real projective space $\mathbf{P}(H_h^{\mathbb{R}})$ as a fiber bundle over $\mathbf{P}(H_h)$ with fibers being isomorphic to S^1 and the projection $p : \mathbf{P}(H_h^{\mathbb{R}}) \mapsto \mathbf{P}(H_h)$. The subset $\tilde{\mathcal{O}} = p^{-1}(\mathcal{O})$ of $\mathbf{P}(H_h^{\mathbb{R}})$ may be identified with $\text{Diff}_+(S^1)$. Let us fix any point x of $\tilde{\mathcal{O}}$. The operators $\tilde{T}_h(g)$ ($g \in \text{Diff}_+(S^1)$) in H_h may be considered as operators in $\mathbf{P}(H_h^{\mathbb{R}})$. The orbit of x under these operators supply $\tilde{\mathcal{O}}$ by another structure of the groupuscular (local group) centered in the point x . The obtained groupuscular structure on $\tilde{\mathcal{O}} \simeq \text{Diff}_+(S^1)$ is not canonical (see [5]), however, the operators of deviation of the new structure from the canonical belong to the Hilbert-Schmidt class \mathcal{HS} . Note though the pullback \tilde{T}_h is monoassociative the noncanonical groupuscular structure on $\text{Diff}_+(S^1)$ is not odular (see [5]) as an artefact of the infinite dimensionality (the image of the exponential map of $\text{Vect}(S^1)$ does not cover any neighbourhood of the identity in $\text{Diff}_+(S^1)$).

Remark 3. The noncanonical groupuscular structure on $\text{Diff}_+(S^1)$ realizes this group as a transformation quasigroup [11] on the orbit \mathcal{O} (a transformation pseudogroup in terms of [12]).

Let us now discuss the complex structure on the orbit \mathcal{O} .

Definition 2. A linear \mathcal{HS} -pseudorepresentation of the Lie group G in the Hilbert space H is a homomorphism of G into $\mathbf{B}(H)/\mathbf{B}_0(H)$, where $\mathbf{B}(H)$ is the group of all invertible bounded operators in H and $\mathbf{B}_0(H)$ is its subgroup of operators A such that $1 - A$ is a Hilbert–Schmidt operator. Two linear representations π_1 and π_2 of the Lie group G in the Hilbert spaces H_1 and H_2 are called *equivalent* iff there exists an isomorphism $S : H_1 \mapsto H_2$ of the linear spaces such that for any element g of the group G there exists an element A_g of $\mathbf{B}_0(H)$ such that

$$\tilde{\pi}_2(g) = A_g S \tilde{\pi}_1(g) S^{-1},$$

where $\tilde{\pi}_i$ are the mappings of G into $\mathbf{B}(H_i)$, which are the pullbacks of the homomorphisms π_i alongside the natural projections $\varepsilon_i : \mathbf{B}(H_i) \mapsto \mathbf{B}(H_i)/\mathbf{B}_0(H_i)$.

This definition may be generalized on any topological semigroup Γ if one change the group $\mathbf{B}(H)$ of all invertible bounded operators to the semigroup $\mathbf{E}(H)$ of all bounded operators in H .

Let $\widetilde{\text{Ner}}$ be the universal covering of the Neretin semigroup, the mantle of the group $\text{Diff}_+(S^1)$ [13]. The semigroup $\widetilde{\text{Ner}}$ is a complex semigroup so one may consider its holomorphic linear \mathcal{HS} -pseudorepresentations.

Theorem 2. *The unitary \mathcal{HS} -pseudorepresentations T_h of $\widetilde{\text{Diff}}_+(S^1)$ in the Hilbert spaces H_h may be extended to the holomorphic linear \mathcal{HS} -pseudorepresentations of the Neretin semigroup Ner .*

Remark 4. The theorem 4 may be generalized to the non-unitarizable Verma modules if one changes the class \mathcal{HS} of the Hilbert-Schmidt operators to the class of all compact operators.

Remark 5. The imbedding $\varepsilon : \mathcal{M} \mapsto \mathbf{P}(H_h)$ is holomorphic.

Remark 5 supplies us by a new construction of the Kirillov's complex structure on $\mathcal{M} = \text{Diff}_+(S^1)/S^1$.

Note that the infinite-dimensional complex manifold \mathcal{M} may be identified with the class S_0 of univalent functions [10] (see also [4,14]). The imbedding $\varepsilon : S_0 \mapsto \mathbf{P}(H_h)$ may be analytically extended to the imbedding $\tilde{\varepsilon}$ of the space $\mathbb{C}_0[[z]]$ of all formal power series of the form $z + c_1 z^2 + c_2 z^3 + \dots + c_n z^{n+1} + \dots$ into the projective space $\mathbf{P}(V_h^{\text{form}})$ over the formal Verma module V_h^{form} for the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ (cf.[14]). Any formal Verma module over $\mathfrak{sl}(2, \mathbb{C})$ is identified with the space of the formal power series $\mathbb{C}[[z]]$ whereas the Verma module itself is identified with the space of polynomials $\mathbb{C}[z]$.

Remark 6 (The hypothetical criterion of the univalence (cf.[14])).

$$\tilde{\varepsilon}^{-1}(\tilde{\varepsilon}(\mathbb{C}_0[[z]]) \cap \mathbf{P}(H_h)) = S_0$$

Corollary (hypothetical). Let $\deg(c_n) = n$, then there exists a universal sequence $\{P_m\}$ of the polynomials $P_m = P_m(c_1, \dots, c_m)$, $\deg(P_m) = m$ such that the formal power series $z + c_1 z^2 + c_2 z^3 + \dots + c_n z^{n+1} + \dots$ defines an univalent function $f(z)$ from the class S_0 iff $\sum_{m=1}^{\infty} |P_m(c_1, \dots, c_m)|^2 < \infty$.

Note that it follows from the results of [14] that there exists a universal sequence $\{P_m^{(k)}\}$ ($1 \leq k \leq p(m)$, $p(m)$ is the partition function) of polynomials $P_m^{(k)} = P_m^{(k)}(c_1, \dots, c_m)$, $\deg(P_m^{(k)}) = m$ such that the formal power series $z + c_1 z^2 + c_2 z^3 + \dots + c_n z^{n+1} + \dots$ defines an univalent function $f(z)$ from the class S_0 iff $\sum_{m=1}^{\infty} \sum_{k=1}^{p(m)} |P_m^{(k)}(c_1, \dots, c_m)|^2 < \infty$. The statement of the corollary is essentially stronger.

2. COMPOSED REPRESENTATIONS OF THE WITT ISOTOPIC PAIR FROM q_R -CONFORMAL STRESS-ENERGY TENSOR AND THE RELATED (q_R -AFFINE) CURRENT

Definition 3A [15] (see also [2:§2.2;16]). The pair (V_1, V_2) of linear spaces is called an (even) isotopic pair iff there are defined two mappings $m_1 : V_2 \otimes \bigwedge^2 V_1 \mapsto V_1$ and $m_2 : V_1 \otimes \bigwedge^2 V_2 \mapsto V_2$ such that the mappings $(X, Y) \mapsto [X, Y]_A = m_1(A, X, Y)$ ($X, Y \in V_1, A \in V_2$) and $(A, B) \mapsto [A, B]_X = m_2(X, A, B)$ ($A, B \in V_2, X \in V_1$) obey the Jacobi identity for all values of a subscript parameter (such operations will be called *isocommutators* and the subscript parameters will be called *isotopic elements*) and are compatible to each other, i.e. the identities

$$\begin{aligned} [X, Y]_{[A, B]_Z} &= \frac{1}{2}([X, Z]_A, Y]_B + [[X, Y]_A, Z]_B + [[Z, Y]_A, X]_B - \\ &\quad [[X, Z]_B, Y]_A - [[X, Y]_B, Z]_A - [[Z, Y]_B, X]_A) \end{aligned}$$

and

$$[A, B]_{[X, Y]_C} = \frac{1}{2}([A, C]_X, B)_Y + [[A, B]_X, C]_Y + [[C, B]_X, A]_Y - \\ [[A, C]_Y, B]_X - [[A, B]_Y, C]_X - [[C, B]_Y, A]_X$$

$(X, Y, Z \in V_1, A, B, C \in V_2)$ hold.

The even isotopic pairs are just the anti-Jordan pairs of J.R.Faulkner and J.C.Ferrar [17] if the characteristics of the basic field is not equal to 2 (see [15;2:§2.2]).

Remark 7. Let M be an arbitrary (smooth) manifold, then the space $\mathcal{O}(M)$ of all smooth functions on M and the space $\text{Vect}(M)$ of all smooth vector fields on M form an isotopic pair with isocommutators

$$[v_1, v_2]_f = \mathcal{L}_{v_1}(f)v_2 - \mathcal{L}_{v_2}(f)v_1 + f[v_1, v_2] \quad (v_1, v_2 \in \text{Vect}(M), f \in \mathcal{O}(M))$$

and

$$[f_1, f_2]_v = \mathcal{L}_v(f_2)f_1 - \mathcal{L}_v(f_1)f_2 \quad (f_1, f_2 \in \mathcal{O}(M), v \in \text{Vect}(M)),$$

here \mathcal{L} denotes the Lie derivative. The constructed isotopic pair is called *the geometric isotopic pair* [15].

Let us consider the complexification of the geometric isotopic pair with $M = S^1$. Let $e_k = ie^{ikt}\partial_t$ and $f_k = ie^{ikt}$ be the natural bases in the spaces $\mathcal{O}^{\mathbb{C}}(S^1)$ and $\mathbb{C}\text{Vect}(S^1)$, t is an angle coordinate on the circle S^1 . The isocommutators have a nice symmetric form [15]:

$$[e_i, e_j]_{f_k} = (i - j)e_{i+j+k}, \quad [f_i, f_j]_{e_k} = (i - j)f_{i+j+k}.$$

The isotopic pair (V_1, V_2) formally generated by the elements e_k ($k \in \mathbb{Z}$) and f_k ($k \in \mathbb{Z}$) will be called *the Witt isotopic pair*.

The isocommutators in (V_1, V_2) are r-matrix, i.e. they may be received from the canonical brackets in the Witt algebra by use of the classical r-matrices [18]. The brackets in the Witt algebra have the form $[e_i, e_j] = (i - j)e_{i+j}$ (and similar for f_k), the classical r-matrices R_x ($[a, b]_x = [R_x(a), b] + [a, R_x(b)]$) have the form $R_{f_k}(e_i) = e_{i+k}$ and $R_{e_k}(f_i) = f_{i+k}$. These r-matrices do not obey the modified Yang-Baxter equation (see [18]). Note that the mapping $x \mapsto R_x$ is multiplicative, i.e. $R_{f_i}R_{f_j} = R_{f_{i+j}}$ and $R_{e_i}R_{e_j} = R_{e_{i+j}}$.

Definition 3B [15]. The pair (V_1, V_2) of linear spaces is called *the (even) isotopic composite* iff there is fixed a set of subpairs (V_1^α, V_2^α) of the pair (V_1, V_2) such that each pair (V_1^α, V_2^α) is supplied by the structure of an (even) isotopic pair. The structures of the isotopic pairs are compatible, it means that the restrictions of the structures for two different α and β on the pair $(V_1^\alpha \cap V_1^\beta, V_2^\alpha \cap V_2^\beta)$ coincide. The isotopic composite is called *dense* iff $\biguplus_\alpha V_1^\alpha = V_1$ and $\biguplus_\alpha V_2^\alpha = V_2$ (here \biguplus denotes the sum of subspaces). The Lie composite is called *connected* iff for all α and β there exists a sequence $\gamma_1, \dots, \gamma_m$ ($\gamma_1 = \alpha, \gamma_m = \beta$) such that $(V_1^{\gamma_l}, V_2^{\gamma_l}) \cap (V_1^{\gamma_{l+1}}, V_2^{\gamma_{l+1}}) \neq \emptyset$.

Example (The Tetrahedron Isotopic Composite). Let us consider a tetrahedron with vertices A, B, C, D and the faces $X = (BCD), Y = (CDA), U = (DBA), V = (BCA)$. The linear spaces V_1 and V_2 are spanned by the generators $e_A, e_B, e_C, e_D, f_X, f_Y, f_U, f_V$ labelled by the vertices and the faces. The subpairs (V_1^α, V_2^α) are spanned by the generators corresponding either to three vertices and a face between them or by three faces and their common vertex. The isotopic pairs (V_1^α, V_2^α) are isotopic pairs associated with the Lie algebra $\mathfrak{so}(3)$ [15,16]. The isocommutators are compatible with the orientation.

The Witt isotopic pair (V_1, V_2) is supplied by the canonical structure of the dense and connected isotopic composite: $V_1^1 = \text{span}(e_i, i \geq -1), V_2^1 = \text{span}(f_i, i \geq 0), V_1^2 = \text{span}(e_i, i \leq 1), V_2^2 = \text{span}(f_i, i \leq 0)$.

Definition 4 [15] (see also [2:§2.2]).

A. A *representation* of the (even) isotopic pair (V_1, V_2) in the linear space H is the pair of linear mappings (T_1, T_2) from the spaces V_1 and V_2 , respectively, to $\text{End}(H)$ such that

$$\begin{aligned} T_1([X, Y]_A) &= T_1(X)T_2(A)T_1(Y) - T_1(Y)T_2(A)T_1(X), \\ T_2([A, B]_X) &= T_2(A)T_1(X)T_2(B) - T_2(B)T_1(X)T_2(A), \end{aligned}$$

where $X, Y \in V_1$ and $A, B \in V_2$.

B. A *representation* of the (even) isotopic composite (V_1, V_2) in the linear space H is the set of representations (T_1^α, T_2^α) of the isotopic pairs (V_1^α, V_2^α) compatible to each other. The compatibility means that the restrictions of the representations for different α and β on the intersections of the corresponding isotopic pairs coincide.

C. Let an isotopic pair (V_1, V_2) is supplied by the structure of the isotopic composite (by the fixing of a set of its proper subpairs), then the representations of the least will be called *the composed representations* of (V_1, V_2) .

Remark 8. All the objects from the definition 4 have their higher combinatorial generalizations (*graph-representations*) – cf.[15].

Remark 9. The representations of the tetrahedron isotopic composite realize the composed representations of the isoquaternionic isotopic pair [15].

Note that the Witt algebra admits a composed representation [1:Topic 9] in the Verma module V_h over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ by the tensor operators of spin 2 (i.e. the q_R -conformal symmetries).

Theorem 3. *The composed representation of the Witt algebra in the Verma module V_h over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ may be extended to the composed representation of the Witt isotopic pair by the tensor operators of spin 1 and 2, namely*

$$\begin{aligned} e_k &\mapsto (\xi + (k+1)h)\partial_z^k \quad (k \geq 0), & e_{-k} &\mapsto z^k \frac{\xi + (k+1)h}{(\xi+2h)\dots(\xi+2h+k-1)} \quad (k \geq 1), \\ f_k &\mapsto \partial_z^k \quad (k \geq 0), & f_{-k} &\mapsto z^k \frac{1}{(\xi+2h)\dots(\xi+2h+k-1)} \quad (k \geq 1), \end{aligned}$$

where $\xi = z\partial_z$.

Note that operators $T_2^1(f_1)$ and $T_2^2(f_{-1})$ generate the Lobachevskii–Berezin algebra (see e.g.[1,3]) whereas $T_1^1(e_k), T_1^2(e_{-k})$ ($k \geq 2$) and $T_1^1(e_0) = T_1^2(e_0)$ generate the nonlinear \mathfrak{sl}_2 in sense of M.Roček [19].

The generating function for the tensor operators of spin 2 is the q_R -conformal stress-energy tensor whereas one for the tensor operators of spin 1 is the q_R -affine current. The q_R -conformal stress-energy tensor may be received from the q_R -affine current (which is involutive [3]) as by the truncated Sugawara construction as an exponential of the associated Fubini-Veneziano field [3].

Problems:

- To globalize the Witt isotopic pair, i.e. to construct the object, whose infinitesimal counterpart is just the Witt isotopic pair. There are hopes that the natural framework for the globalization may be found in lines of [20] (“the universal quantum group”). The result should be described as a quantum group with the multiple noncommutative parameters q_i^* , which form also a quantum group with elements of the first one as the multiple noncommutative parameters of quantization.
- To globalize the construction of the composed representation of the Witt composite in the Verma modules V_h over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.
- To describe an algebraic structure analogous to the Witt isotopic composite and represented by tensor operators of the higher spins.

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* The idea to consider the multiparametric quantum deformations was explored by Yu.I.Manin and his pupils as I know. However, the multiple parameters of their deformations always commute.

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